DATA PROCESSING OF STEADY-STATE THERMOPHYSICAL EXPERIMENTS WHILE TAKING INTO ACCOUNT THE ERROR IN ALL MEASURED QUANTITIES

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An algorithm for the determination of the functional dependence of temperature on a spatial coordinate and of thermal conductivity and the linear thermal expansion coefficient is proposed.

INTRODUCTION

In recent years various modifications of the least-squares method have been widely used to analyze experimental data [1-4]. Some treatments emphasize the development of the least-squares method for treating nonlinear problems, in which a priori information regarding the covariant matrix is used in order to obtain a stable solution [2].

The second approach, the main result of which was described by Lybanon [4], emphasizes account for the error in all measured quantities as its principal goal. The traditional least-squares method, which is often used in practice, is implicitly based on the hypothesis that the observed quantity is measured without an error. This hypothesis leads to biased estimates of the determined parameters [3]. Even though the main ideas behind this approach were analyzed between 1969 and 1982 in a number of articles [3-6], it was not widely put into practice for analyzing thermophysical measurement data and for solving the problem of determinacy.

The goals of the present work include the analysis and numerical implementation of such an algorithm for analyzing experimental data in which the errors in measuring both temperature and a spatial coordinate are taken into account. The problem is examined on the example of an experimental investigation of thermal conductivity by the stationary radial heat flow method, also known as the cylinder method [7].

1. Data Analysis Algorithm. In considering the placement of temperature-sensitive elements on the sample, one must take into account the labor intensity of this process as well as the distorting influence of the measuring instrument on the temperature field. For those reasons it is most efficient to adopt such variants of the measurement scheme in which the number of thermocouples is the minimum number consistent with the required reliability of the obtained results.

It is possible to increase the accuracy of determining the desired characteristics either by increasing the number of parameters of single-type functions (for example, polynomials) or by expanding the function set while retaining a limited number of parameters.

The first method is used primarily when there is a large number of measurements. Here, nevertheless, low-order polynomials which are as a rule not higher than fifth or sixth order are used. Increasing the polynomial order any further might lead to ill-conditioned systems of normal equations and, ultimately, to a loss of accuracy.

The second method may be efficiently used with comparatively small amounts of experimental data, which is the case in the stationary cylinder and plate methods.

In order to solve the posed problem of finding temperature fields, we propose to use a set of reciprocal functions, such as those given in Table 1. The possibility of using both the direct and the inverse functions to describe the recovered interdependence of temperature and spatial coordinates is conditioned upon the strict monotonic character of T(r) [or r(T)] for the majority of investigated data. We note that the number of functions

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TABLE 1. Reciprocal Functions

Ordinal no.	$r(a_0, a_1, T)$	$T(a_0, a_1, r)$	Ordinal no.	$r(a_0, a_1, T)$	$T(a_0, a_1, r)$	Ordinal no.	$r(a_0, a_1, T)$	$T(a_0, a_1, r)$
1	$a_0 + a_1T^n$	$\left[\frac{r - a_0}{a_1}\right]^{\frac{1}{n}}$	9	$a_0 m^{a_1 T^n}$	$\left[\frac{\ln(r/a_0)}{a_1 \ln m}\right]^{\frac{1}{n}}$	16	$\frac{1}{C}\left[\ln\left(\frac{T - a_0}{a_1}\right)\right]^{\frac{1}{n}}$	$a_0 + a_1 \exp((rC)^n)$
2	$\frac{1}{a_0 + a_1T^n}$	$\left[\left(\frac{1}{r} - a_0\right)\Big/ a_1\right]^{\frac{1}{n}}$	10	$a_0 + a_1(\ln T)^n$	$\exp\left[\left(\frac{r - a_0}{a_1}\right)^{\frac{1}{n}}\right]$	17	$\frac{1}{C}\left[\ln\left(\frac{a_1}{T - a_0}\right)\right]^{\frac{1}{n}}$	$a_0 + a_1 \exp(-(rC)^n)$
3	$\frac{T^n}{a_0 + a_1T^n}$	$\left[\frac{a_0 r}{1 - a_1 r}\right]^{\frac{1}{n}}$	11	$[a_0 + a_1(\ln T)^n]^{-1}$	$\exp\left[\left(\frac{1 - a_0 r}{a_1 r}\right)^{\frac{1}{n}}\right]$	18	$\frac{1}{C}\left[\ln\left(\frac{1 - a_0T}{a_1T}\right)\right]^{\frac{1}{n}}$	$\frac{1}{a_0 + a_1 \exp((rC)^n)}$
4	$a_0 a_1^{T^n}$	$\left[\frac{\ln(r/a_0)}{\ln a_1}\right]^{\frac{1}{n}}$	12	$\left[\frac{T - a_0}{a_1}\right]^{\frac{1}{n}}$	$a_0 + a_1 r^n$	19	$\frac{1}{C}\left[\ln\left(\frac{a_1T}{1 - a_0T}\right)\right]^{\frac{1}{n}}$	$\frac{1}{a_0 + a_1 \exp(-(Cr)^n)}$
5	$a_0 + a_1 \exp((bT)^n)$	$\frac{1}{b}\left[\ln\left(\frac{r - a_0}{a_1}\right)\right]^{\frac{1}{n}}$	13	$\left[\left(\frac{1}{T} - a_0\right)\Big/ a_1\right]^{\frac{1}{n}}$	$\frac{1}{a_0 + a_1 r^n}$	20	$\left[\frac{\ln(T/a_0)}{a_1 \ln m}\right]^{\frac{1}{n}}$	$a_1 m^{a_1 r^n}$
6	$a_0 + a_1 \exp(-(bT)^n)$	$\frac{1}{b}\left[\ln\left(\frac{a_1}{r - a_0}\right)\right]^{\frac{1}{n}}$	14	$\left[\frac{a_0T}{1 - a_1T}\right]^{\frac{1}{n}}$	$\frac{r^n}{a_0 + a_1 r^n}$	21	$\exp\left[\left(\frac{T - a_0}{a_1}\right)^{\frac{1}{n}}\right]$	$a_0 + a_1(\ln r)^n$
7	$\frac{1}{a_0 + a_1 \exp((bT)^n)}$	$\frac{1}{b}\left[\ln\left(\frac{1 - a_0 r}{a_1 r}\right)\right]^{\frac{1}{n}}$	15	$\left[\frac{\ln(T/a_0)}{\ln a_1}\right]^{\frac{1}{n}}$	$a_0 a_1^{T^n}$	22	$\exp\left[\left(\frac{1 - a_0T}{a_1T}\right)^{\frac{1}{n}}\right]$	$[c_0 + a_1(\ln r)^n]^{-1}$
8	$\frac{1}{a_0 + a_1 \exp(-(bT)^n)}$	$\frac{1}{b}\left[\ln\left(\frac{a_1 r}{1 - a_0 r}\right)\right]^{\frac{1}{n}}$						

TABLE 2. Recovery of the Thermal Linear Expansion Coefficient

Experimental values				
d_i, m	1,000	1,002	1,004	1,006
T_i, degrees	1,2	202,0	397,0	590,0

Method	Formula for the temp., T dependence on sample length d	Coeff. $\alpha \times 10^{-6}$, 1/deg	Relative error, %	
			δ_{Td}	δ_α
Least-squares method	$T = 3{,}574 + 98361 \ln d$	10,17	75	1,7
Modified method	$T = 1{,}1 + 100046{,}99 \ln d$	9,9953	6	0,05
Exact dependence	$T = 1{,}0 + 100000{,}0 \ln d$	10,00	0	0

in Table 1 is not limited to twenty-two. For example, for n = −2, −1, 1, 2; m = 0, 1, e, 5, 10; b = 10^{-3}, $2\cdot10^{-3}$; C = 10^3, $2\cdot10^3$, the general number of variants is equal to 148. The main idea behind the modification of the least-squares method (also known as the effective dispersion method [3]) is based on the minimization of the following sum:

$$S = \sum_{i=1}^{N} [W_{ri}(r_i - r(a_0, \ldots, a_k, T_i))^2 + W_{T_i}(T_i - T(a_0, \ldots, a_k, r_i))^2], \tag{1}$$

where W_{r_i} and W_{T_i} are assigned a priori values. The principal difficulty of this method — as well as that developed by Lybanon [4] — is in finding the correct method for determining W_{r_i} and W_{T_i}.

An alternative variant for finding the best function for the table of experimental data is based on the minimization of the sum of squares of relative errors in the temperature and the radial coordinate:

$$\chi(a_0, \ldots, a_k) = \sum_{i=1}^{N} \{[(r_i - r(a_0, \ldots, a_k, T_i))/r_i]^2 + [(T_i - T(a_0, \ldots, a_k, r_i))/T_i]^2\}. \tag{2}$$

Here, in order to take into account the errors of all the measured quantities, it is not necessary to assign or calculated, by a complex method, weight coefficients which stand in front of the squares of the differences between the tabulated and calculated values. The roles of such coefficients are played by the table values of r_i and T_i. Jeffreys [3] suggested that the minimization of the nonlinear functional, Eq. (1), may not always be advantageous due to a poor choice of initial approximations for the unknown parameters or due to the divergence of the Newton method [8].

In order to avoid such difficulties in using the algorithm for the minimization of the sum of squares of relative errors, Eq. (2), we propose to use reciprocal functions with one linearizable member.

The procedure for minimizing Eq. (2) is as follows. At the beginning, for the first functions $T(a_0, \ldots, a_k, r)$ or $r(a_0, \ldots, a_k, T)$ chosen from Table 1, we determine $a_0^{(0)}, \ldots, a_k^{(0)}$ by using the usual least-squares method. Next, using the standard procedure — for example, the gradient descent method — we minimize the total sum of squares of relative errors, Eq. (2); as initial approximations for the unknown parameters, we propose to use the already found $a_0^{(0)}, \ldots, a_k^{(0)}$ After each step in the approach to the minimum (or after a small number of such steps), the following conditions are checked: Is the limit on the number of steps in the approach toward the desired minimum exceeded? Does the value of the root-mean-square of the sum of relative errors $\sqrt{\chi(a_0^{(j)}, \ldots, a_k^{(j)}/(2N)}$ approach an a priori assigned level corresponding to the instrumental error:

TABLE 3. Temperature Dependence of Thermal Conductivity for the Temperature Fields in Table 1

Function no. from Table 1	$\lambda(T)$
1	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1} \left(\frac{a_0 + a_1 T_H^n}{a_0 + a_1 T^n}\right)$
2	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1} \left(\frac{a_0 + a_1 T_H^n}{a_0 + a_1 T^n}\right)$
3	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1} \left(\frac{a_0 + a_1 T_H^n}{a_0 + a_1 T^n}\right)$
4	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1}$
5	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1} \frac{\exp(b(T^n - T_H^n))(a_0 + a_1 \exp((bT)^n))}{(a_0 + a_1 \exp((bT)^n))}$
6	$\frac{\lambda_H (T)^{n-1} \exp((b(T_H^n - T^n)))(a_0 + a_1 \exp(-(bT)^n))}{T_H^{n-1}(a_0 + a_1 \exp(-(bT)^n))}$
7	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1} \exp(b(T^n - T_H^n)) \frac{(a_0 + a_1 \exp((bT_H)^n))}{(a_0 + a_1 \exp((bT)^n))}$
8	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1} \exp(-2b(T^n - T_H^n)) \frac{(a_0 + a_1 \exp(-(bT)^n))}{a_0 + a_1 \exp(-(bT)^n)}$
9	$\lambda_H \left(\frac{T}{T_H}\right)^{n-1}$
10	$\lambda_H \left(\frac{\ln T}{\ln T_H}\right)^{n-1} \frac{T_H (a_0 + a_1 (\ln T_H)^n)}{T(a_0 + a_1 (\ln T)^n)}$
11	$\lambda_H \left(\frac{\ln T}{\ln T_H}\right) \left(\frac{T_H}{T}\right) \frac{(a_0 + a_1 (\ln T)^n)}{(a_0 + a_1 (\ln T_H)^n)}$
12	$\frac{\lambda_H (T_H - a_0)}{T - a_0}$
13	$\lambda_H \left(\frac{a_0 T_H - 1}{a_0 T - 1}\right) \frac{T_H}{T}$
14	$\lambda_H \left(\frac{T_H}{T}\right) \left(\frac{a_1 T_H - 1}{a_1 T - 1}\right)$
15	$\lambda_H \frac{T_H \ln(T_H/a_0)}{T \ln(T/a_0)}$
16	$\lambda_H \frac{(T_H - a_0)\ln((T_H - a_0)/a_1)}{(T - a_0)\ln((T - a_0)/a_1)}$
17	$\frac{\lambda_H (T_H - a_0)\ln(a_1/(T_H - a_0))}{(T - a_0)\ln(a_1/(T - a_0))}$
18	$\frac{\lambda_H (a_0 T_H - 1) T_H \ln((1 - a_0 T_H)/(a_1 T_H))}{(a_0 T - 1) T \ln((1 - a_0 T)/(a_1 T))}$
19	$\frac{\lambda_H (1 - a_0 T_H) T_H \ln((a_1 T_H)/(1 - a_0 T_H))}{(1 - a_0 T) T \ln((a_1 T)/(1 - a_0 T))}$
20	$\lambda_H \left(\frac{T}{T_H}\right) \frac{\ln(T_H/a_0)}{\ln(T/a_0)}$
21	$\lambda_H \left(\frac{T_H - a_0}{T - a_0}\right)^{\frac{n-1}{n}}$
22	$\frac{\lambda_H T^2 ((1 - a_0 T_H)/(a_1 T_H))^{(1-n)/n}}{T_H^2 ((1 - a_0 T)/(a_1 T))^{(1-n)/n}}$

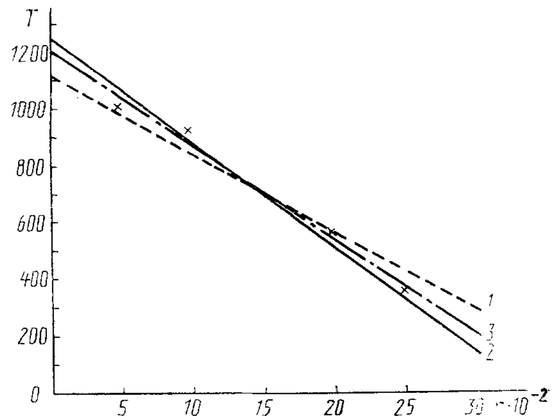

Fig. 1

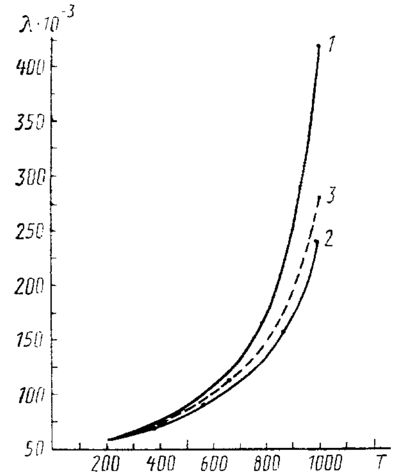

Fig. 2

Fig. 1. Recovery of a functional relation between temperature and a spatial coordinate: 1) the result of using the traditional least-squares method: T = 1133 − 28.6r; 2) the result of minimizing the sum of squares of relative errors in measuring the temperature and the spatial coordinate: T = 1270 − 37.7r; 3) the exact dependence: T = 1214 − 34r. Points represent the data used to construct the temperature field. T in °C, r is in meters.

Fig. 2. Temperature dependence of thermal conductivity: 1) the result of using the traditional least-squares method: λ(T) = 56/(1133 − T); 2) the result of minimizing the sum of squares of relative errors in temperature and spatial coordinate measurements: λ(T) = 64/(1270 − T); 3) the exact dependence λ(T) = 61/(1214 − T). λ, W/(m·K).

$$\sqrt{\chi(a_0^{(j)},\ \dots,\ a_k^{(j)})/(2N)} \approx \sqrt{0.5\,(\delta^2_{\mathrm{instr\,T}}+\delta^2_{\mathrm{instr\,r}})} = \delta_{\mathrm{instr\,Tr}}. \tag{3}$$

We note that the relation, Eq. (3), corresponds to the generally accepted solution to the problem of finding the method for choosing the best approximation [9, 10].

If the number of gradient descent steps is exhausted, or the minimum of $\chi(a_0,\ \dots,\ a_k)$ is attained, or the condition in Eq. (3) is first violated, then the output values of $a_0^{(j)}$ and $a_k^{(j)}$ are stored together with the aforementioned root-mean-square relative errors, and a transition is made to the next pair of functions in Table 1. At the final stage the calculated errors $\sqrt{\chi/(2N)}$ are compared one at a time with $\delta_{\mathrm{instr\,Tr}}$. From all the functions only those for which the left-hand side of Eq. (3) is minimum are chosen.

Using the discovered relation for the change of sample length with temperature, it is possible to set up an analytic expression for the thermal linear expansion coefficient α(T):

$$\alpha(T) = (1/d(T))|\, dd/dT. \tag{4}$$

Table 2 shows the result of reconstructing α(T) while accounting for the error in measuring r_i and T_i as well as for the case when only the error in measuring the temperature is considered.

Analogous to the temperature field, the dependence of thermal conductivity λ on temperature T is recovered by employing the one-dimensional steady-state thermal conductivity model which is implemented in the cylinder method:

$$\left(\frac{1}{r}\right)\frac{d}{dr}\,[r\lambda(T)\,dT/dr] = 0, \tag{5}$$

$$\lambda(T_{\text{н}}) = \lambda_{\text{н}}. \tag{6}$$

For each pair of functions T(r) and r(T), we selected from Table 1 a function λ(T) which satisfies Eqs. (5) and (6). Functions λ(T) found by this method are shown in Table 3. After the analysis of the measured (r_i, T_i) values by comparing the calculated and instrumental errors as described above, a triplet of functions T(r), r(T), and λ(T) that are connected by Eq. (5) and condition (6) are chosen.

We also studied another method for recovering thermal conductivity. For this purpose we select a function r(T) that is the inverse of a function T(r) which satisfies Eq. (5) and is subject to the following boundary conditions:

$$T(r_1) = T_1,\ T(r_N) = T_N. \tag{7}$$

Here, $T_H = T_N$. The sought dependence has the form

$$r(T) = r_N \exp\left[(\ln(r_1/r_N)) \int_{T_N}^{T} \lambda(T)\,dT \Big/ \int_{T_N}^{T_1} \lambda(T)\,dT\right]. \tag{8}$$

The function λ(T) is initially assigned a linear $\lambda_N + 2a_0(T - T_N)$, a power $\lambda_N(T/T_n)^{a_0}$, an exponential $\lambda_N \exp[(a_0(T - T_N)]$, or a logarithmic $\lambda_N + a_0 \ln(T/T_N)$ form which depends on one unknown parameter. With the help of Eq. (8), both the direct and inverse functions connecting the temperature and the spatial coordinate may be obtained for each numerated dependence. The unknown parameter a_0 is found by minimizing the error, Eq. (2), and by subsequently checking the correspondence between the calculated and instrumental errors.

Whereas the first algorithm minimizes with respect to all N measured (r_i, T_i) pairs, the second algorithm fixes (r_1, T_1) and (r_N, T_N) as boundary conditions; in fact, only N − 2 points are used for the minimization. When, for example, only four thermocouples are arranged on a sample, the class of desired functions λ(T) narrows down to one-parameter functions, and fixing the boundaries of the temperature field is equivalent to interpolating at the boundaries without diminishing possible error jumps at these points. These facts attest to a deficiency of the latter algorithm.

CONCLUSIONS

The algorithms discussed in this work were implemented on a computer. The results of recovering the thermal linear expansion coefficient is shown in Table 2. Figure 1 shows the results of reconstructing functions T(r). Using these temperature fields, we calculated functions λ(T) shown in Fig. 2.

The developed codes for analyzing experimental data may be used in measuring thermal conductivity by the cylinder and the plate methods.

NOTATION

T, temperature; r, radial coordinate; λ, thermal conductivity; α, thermal linear expansion coefficient; N, the number of thermally-sensing elements in the measured sample; λ_H, thermal conductivity at temperature T_H; d(T), the linear dimension of the investigated sample at temperature T; $a_0, \ldots, a_k$, reconstruction parameters; $\delta_{\text{instr}\,r}$, $\delta_{\text{instr}\,T}$, instrumental relative errors in the measurement of the spatial coordinate r and temperature T, respectively.

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